Pebbling a Chess board

Answers

7 8 9 10 **11** 12

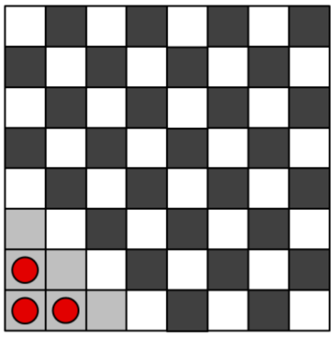
TI-Nspire™ Activity Student 30 min

Comments

- A PowerPoint file is included with this activity that can be used to help illustrate the problem to the class; the same file also contains the solution process.
- A quick web search will identify numerous websites that contain versions of the Pebbling the Chessboard challenge, some allow the user to define the number of starting pieces and the shaded region making them particularly useful for the last question in this activity.

Introduction

This challenge is based on the mathematics problem created by Maxim Kontsevich, a very talented French and Russian mathematician. The problem involves getting three discs out of a specific region of a chess board. In this example we will consider the region to consist of the six shaded squares in the bottom left corner of the board. The rules of the challenge are as follows:



- Discs can only move forward (top of page) one space at a time.
- Each time a disc is moved it divides into two; one piece going forward and the other immediately to the right.
- Discs cannot be placed on top of one another.

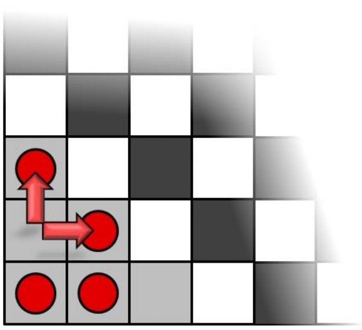
How many moves does it take to get all three pieces out of the shaded region?

Equipment

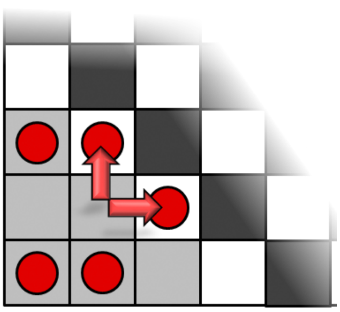
- TI-Nspire Calculator
- Chess Board & Counters[1]

Investigation

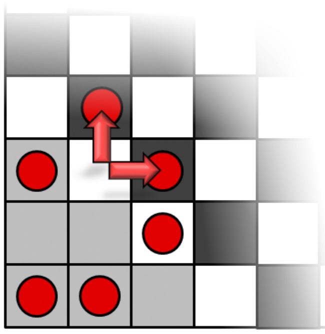
A possible first move.



A second move …



One more move and the first disc will be clear of the shaded region.



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[1] Numerous websites host a version of this challenge.

Author: P. Fox

Question: 1.

What is the minimum number of moves required to leave just two discs in the shaded region?

Four moves.
(One more move in the above diagram will see only two discs remaining in the shaded region.)

Question: 2.

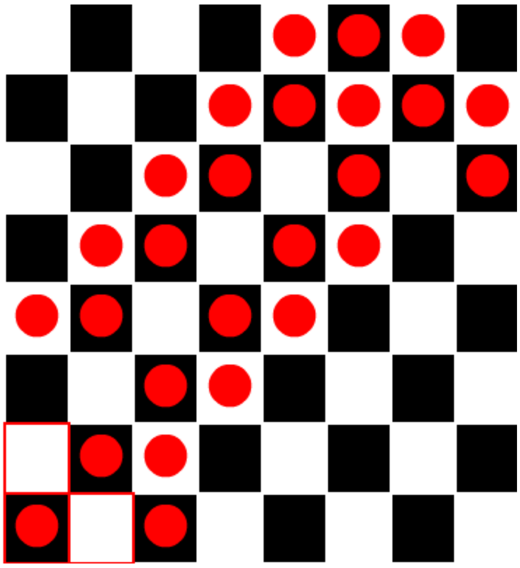
Estimate the number of moves required so that only one disc remains in the shaded region?

Answers will vary… the question is designed to get a sense of whether students think the problem is indeed 'solvable'.

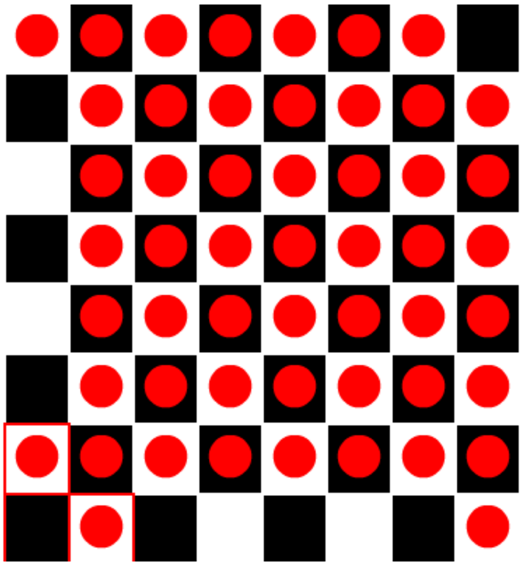
Question: 3.

Using either a physical or digital platform, explore the problem and see if it is possible, within the confines of a chess board, to leave just one disc in the shaded region. Record your best attempt.

This sample uses the strategy that sufficient spaces need to be cleared around the original discs. The result is an upward – right progression where it soon becomes apparent this pattern could go on forever and not leave enough space.

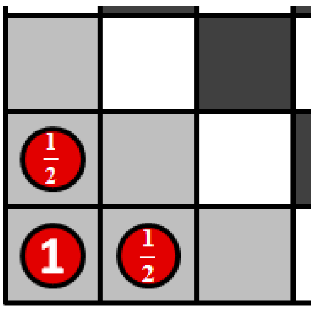


This sample focuses on a numerical strategy. (Refer Question 4.) The board can almost be filled, a necessity in the numerical strategy, however some regions are not obtainable which further supports the notion that the problem cannot be solved.



The problem becomes frustratingly difficult to free just two discs; however failure to succeed after numerous attempts does not mean a solution does not exist. It is possible to explore this problem theoretically using a numerical approach rather than relying on trial and error.

Suppose each counter is assigned a value. Consider the counters as shown:



Each time a counter is moved and splits into two new counters, the new counters are each worth half the original value. So when the very first half point counter is moved, the two new counters are worth a quarter of a point each.

Author: P. Fox

Question: 4.

What is the total value of the counters at the start of the problem?

2 = 1 + ½ + ½

Question: 5.

After one move, what is the total value of the counters in the problem? 2 = 1 + ½ + ¼ + ¼ (Same as before – Leading to the notion that the **sum** is 'invariant'.)

Question: 6.

After two moves, what is the total value of all the counters in the problem?

There are several options here, but all of them give the sum: 2, once again leading to the notion that the **sum** is invariant.

Question: 7.

Imagine all the discs are now outside the grey area. What would be the sum of all the discs?

The sum would still be equal to 2. The sum is invariant because the point value is always distributed to the two new discs created on each move. If all the discs are outside the grey area (problem solved) the sum of the discs outside the grey area must be 2 as there would be zero points inside the grey area.

Question: 8.

Start the problem from the beginning and record in each square, the value of each counter in each corresponding square as the problem progresses.
Record your results on a blank chess board.

$\frac{1}{128}$	$\frac{1}{256}$	$\frac{1}{512}$	$\frac{1}{1024}$	$\frac{1}{2048}$	$\frac{1}{4096}$	$\frac{1}{8192}$	$\frac{1}{16384}$
$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{256}$	$\frac{1}{512}$	$\frac{1}{1024}$	$\frac{1}{2048}$	$\frac{1}{4096}$	$\frac{1}{8192}$
$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{256}$	$\frac{1}{512}$	$\frac{1}{1024}$	$\frac{1}{2048}$	$\frac{1}{4096}$
$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{256}$	$\frac{1}{512}$	$\frac{1}{1024}$	$\frac{1}{2048}$
$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{256}$	$\frac{1}{512}$	$\frac{1}{1024}$
$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{256}$	$\frac{1}{512}$
$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{256}$
1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$

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Question: 9.

Determine the sum of the values in the first 8 rows, **excluding** the shaded region, and also the sum of the values in the first 8 rows if the board could be extended indefinitely, **excluding** the shaded area.

Comments:

The terms on each square of the board form a geometric sequence with a common ratio of ½.

Students could use the formula: $S_n = \dfrac{a(1-r^n)}{1-r}$ where the common ratio is ½, n = number of squares to be added and a is the first eligible term in each row. Alternatively, students can use the calculator:

First row using the calculator: $\sum_{n=3}^{7} \frac{1}{2^n} = \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} = \frac{31}{128}$

Using the formula $a\left(\dfrac{1-r^n}{1-r}\right)$ where $a = \frac{1}{8}$, $r = \frac{1}{2}$ and $n = 5$ then $\frac{1}{8}\left(\dfrac{1-\left(\frac{1}{2}\right)^5}{1-\left(\frac{1}{2}\right)}\right) = \frac{31}{128}$

Row	Sum (Outside grey area – on board)	Sum (Indefinitely)
Row 1 (Bottom Row)	Sum = $\frac{31}{128}$	$\frac{1}{4}$
Row 2:	Sum = $\frac{63}{256}$	$\frac{1}{4}$
Row 3:	Sum = $\frac{127}{512}$	$\frac{1}{4}$
Row 4:	Sum = $\frac{255}{1024}$	$\frac{1}{4}$
Row 5:	Sum = $\frac{255}{2048}$	$\frac{1}{8}$
Row 6:	Sum = $\frac{255}{4096}$	$\frac{1}{16}$
Row 7:	Sum = $\frac{255}{8192}$	$\frac{1}{32}$
Row 8:	Sum = $\frac{255}{16384}$	$\frac{1}{64}$

Question: 10.

Determine the total value of the squares on the chess board outside the shaded region and comment on the result. **Hint**: What does this total need to be in order for the grey region to be empty?

The sum of all the squares on the board can be determined using the sum of the column … however there are other ways such as using the diagonals: $\sum_{n=3}^{7} \frac{n+1}{2^n} + \sum_{n=8}^{14} \frac{15-n}{2^n} = \frac{19969}{16384} \approx 1.2188$

Author: P. Fox

Given the total sum of all the discs at any one time is invariant, the total outside the grey area needs to be equal to 2, therefore the problem on the chess board only is not possible, even if every square could be occupied by a disc.

Question: 11.

The problem may be easier if the board were made indefinitely large. Determine the total value of the squares outside the shaded region on a board of infinite size and comment on the result.

This problem is actually easier than summing the squares on the board. The first three rows extended indefinitely result in the sum to infinity: $\frac{a}{1-r}=\frac{\frac{1}{8}}{1-\frac{1}{2}}=\frac{1}{4}$.

From here students can either repeat the summation with a different starting value, or realise that each subsequent row is just missing the first term… therefore successively subtracting this quantity from the sum will also work. Students should then realise that 'sums' for each row also form a geometric sequence and can therefore be totalled in the same way resulting in a total sum: 1.25, still well short of the total required.

Question: 12.

If the shaded region were made smaller so that it was only just big enough to contain the original three discs and the board made infinitely large; the problem would appear much easier. Determine the number of moves required to remove all three discs from the new shaded region.

If the grey area exists only over the three initial discs the total value of the discs is: 2 and the area outside also equal to 2. This would however also imply that the problem could never be 'solved' as you would need to continue moving discs forever. Furthermore, if just one space was left empty, then a solution would not exist. It is true that groups of spaces cannot be occupied; therefore we are still a long way from a solution.

Question: 13.

Returning to the original chess board sized problem, imagine only two discs existed on the board at the start of the problem. Design a new, connected shaded region that would provide a challenge and would be solvable.

Answers will vary … the minimum requirement is for the sum of the squares outside the 'grey' region to be greater than that inside the region, the greater the difference the more likely it is for the problem to be solved. It is not possible to completely fill the board so a difference must exist.

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