Pythagoras – Episode 3

Teachers Teaching with Technology™
Professional Development from Texas Instruments

Teacher Notes & Answers

7 8 **9** 10 11 12

TI-Nspire | Investigation | Student | 30 min

Introduction

The first two stages in this journey have probably raised as many questions as they have answered. Pythagoras's theorem has thrown us into a numerical world that did not previously exist; numbers that cannot be fully tamed or expressed have a strangle hold on our synapses.

In *order* to proceed, we must *take* the **lead** and look **in fine detail** at the information that has been highlighted. It is in this clue that our fractional knowledge of the rational will transition to irrational. Once you have cracked the clue the *code will be revealed* and the next stage of your journey will become clearer.

Cryptic Clue:

"In *order* to proceed"**,** (need to solve a clue first) and cryptically "order" has something to do with the answer.
"we must *take* the **lead**", take or more explicitly, take away, is to subtract. The word "lead" is highlighted and is most likely the thing to be taken away or removed.
"look **in fine detail** at the information that has been highlighted." The words "in fine detail" have been highlighted, we need to look into the find detail … and subtract 'lead'.
The highlighted letters are: I N F I N E D E T A I L, subtracting the letters LEAD leaves: I N F I N E T I
Rearrange the *order* of these letters (anagram): INFINITE
Rational numbers can be expressed as a ratio between two whole numbers. The next section of this investigation looks at how finite continued fractions can be created from a fraction (rational) but infinite continued fractions can represent irrational numbers; so the link between rational and irrational is infinite.
The clue continues … "once you have cracked the clue the *code* will be revealed …".
When students store "infinite" (must be in quotation marks) in the variable "clue", a QR code will be displayed on the screen linking them to a video about "continued fractions".

Teacher Notes:

Students can choose to watch the video, it is not critical, but it does create a richer overall experience.
The video progresses quite fast, particularly the first part where an improper fraction is changed to a continued fraction. One of the best things about videos is that you can pause and rewind, unlike most teacher explanations in the classroom. This aspect of YouTube makes it a wonderful learning opportunity for students wanting to seek help.
Students who watch the video should recognise question 3 and the continued fraction expressed on page 1.4.
The presenter: BlackPenRedPen (Steve Chow) has many videos, most of them at a level well above middle school mathematics, however there are many others, just like Steve, that produce support content for free.

Author: P. Fox

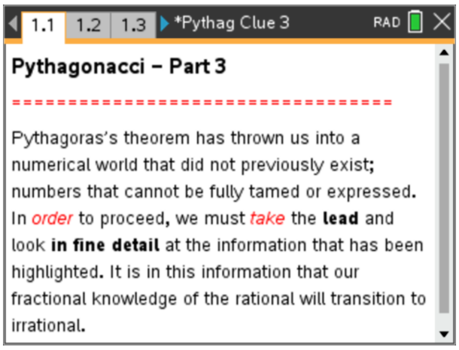
Continued Fractions

Open the TI-Nspire file: **Pythag Clue 3**

The first page of the document contains some important text from above. Read it carefully! If you can figure out the answer, store it on page 1.2 where it currently says clue:= "empty".

If you cannot solve the riddle, you can still progress to page 1.3.

If you have an answer to the riddle, it must be placed in quotation marks as per the sample provided on page 1.2. The simple way to do this is to copy and paste the sample.



Page 1.3 contains a calculator application, ready for some calculations.

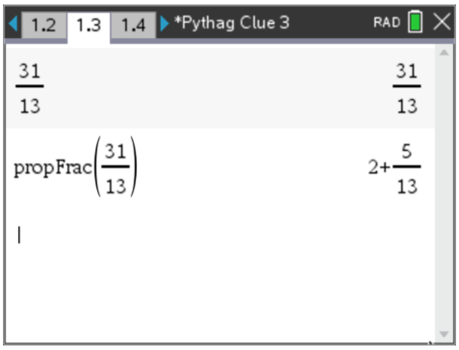
The fraction key on TI-Nspire can be accessed by pressing: ctrl + ÷

By default, a calculation such as 31 ÷ 13 will be expressed as an improper fraction. [See opposite]

This can be changed to a mixed fraction by pressing:

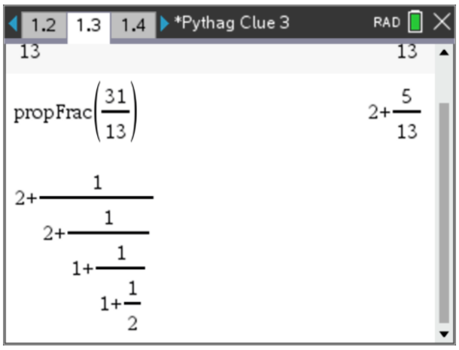
menu > **Number** > **Fraction Tools** > **Proper Fraction**

The previous calculation can be copied and pasted into the propFrac command or simply re-typed.



This is not the only way to represent a fraction in mathematics.
A continued fraction provides a different way of looking at the 'remainder' portion: 5 ÷ 13.

Use your calculator to write the fraction shown below and opposite).
You will need to use the fraction template: ctrl + ÷ and the tab or arrow keys to navigate.



$$2+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2}}}}$$

Question: 1.

In reference to the continued fraction shown above:

a) What answer does your calculator produce?

Answer: 31/13

b) If you had to do this question by hand (no calculator), where would the first calculation be?

Answer: 1 + ½ = 3/2 [Students may highlight the original continued fraction]

c) If you had to do this question by hand (no calculator), without actually doing the calculation, where would the second calculation occur?

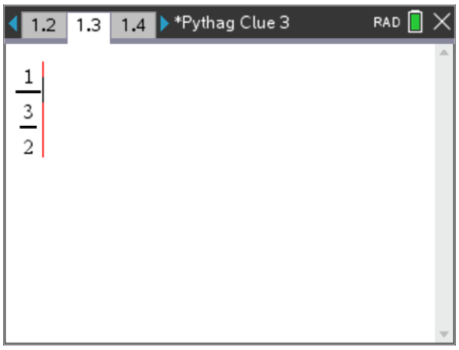
Answer: 1 + 1/(previous answer)

Author: P. Fox

To help understand how these continued fractions are created, we first need to understand why the sequence of '1's appears in the numerator of each 'sub fraction'.

Express the fraction shown opposite on your calculator, notice that the vinculum (line separating the numerator and denominator) is larger at the top. This means the expression is interpreted as: 1 ÷ (3 ÷ 2).

Try some examples of your own, each must be of the form: 1 ÷ (a ÷ b).



Question: 2.

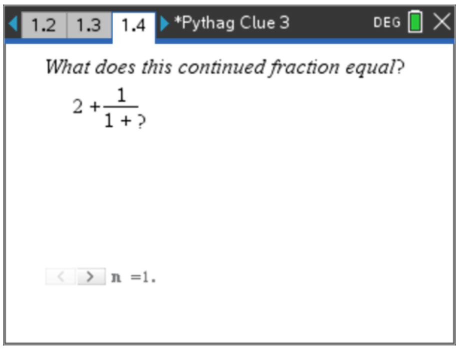
Based on your answers to the previous calculator task, express the following as simple fractions:

a) $\dfrac{1}{\dfrac{5}{6}}$ b) $\dfrac{1}{\dfrac{3}{7}}$ c) $\dfrac{1}{\dfrac{13}{121}}$

Answers: a) $\frac{6}{5}$ or $1\frac{1}{5}$, b) $\frac{7}{3}$ or $2\frac{1}{3}$, c) $\frac{121}{13}$ or $9\frac{4}{13}$

Navigate to page 1.4 and use the slider (spinner) to progressively reveal or hide sections of the fraction. The ability to hide will be useful when performing the calculations by hand.

$$2+\cfrac{1}{1+\cfrac{1}{5+\cfrac{1}{2}}}$$



Question: 3.

Use spinner on page 1.4 to reveal the entire continued fraction, then calculate each portion, one at a time.

a) What calculation needs to be performed first? Express your calculation as an improper fraction.

Answer: First calculation: $5+\frac{1}{2}=\frac{11}{2}$

b) What calculation needs to be performed second? Express your calculation as an improper fraction.
Note: You will need your answer from part (a) first. Consider adjusting the slider accordingly.

Answer: Second calculation: $1+\frac{2}{11}=\frac{13}{11}$

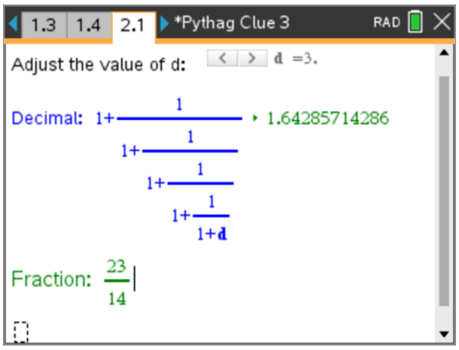
c) What calculation needs to be performed last? Express your calculation as an improper fraction.
Note: You will need your answer from parts (a & b) first. Consider adjusting the slider accordingly.

Answer: Third calculation: $2+\frac{11}{13}=\frac{37}{13}$ [Same as the video]

Teacher Notes: At this stage, we are mostly interested in reversing out of a continued fraction, however, there is merit in going forward and producing continued fractions, as per the video. Another skill that is introduced/reviewed here is the notion of 'flipping' a fraction through division. That is: 4 ÷ ½ is the same as 4 x 2.

Author: P. Fox

Navigate to page 2.1, this is a Notes application. A continued fraction is displayed. The value of (d) is displayed at the top of the screen. Changing this value will automatically generate both the decimal and fraction answers for the continued fraction displayed.



If the values for the calculations are not displayed in the Notes application, press menu > **Actions** > **Activate All**

Question: 4.

Use the slider to change the value of (d) and record the results for each value, decimal and fraction.

d	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
Decimal:	1.6	1.625	1.6364..	1.6429..	1.6471..	1.6500	1.6522	1.6538	1.6551	1.6563
Fraction:	$\frac{8}{5}$	$\frac{13}{8}$	$\frac{18}{11}$	$\frac{23}{14}$	$\frac{28}{17}$	$\frac{33}{20}$	$\frac{38}{23}$	$\frac{43}{26}$	$\frac{48}{29}$	$\frac{53}{32}$

What do you notice about the decimal values?

Answer: The value for d effects only the second decimal place and appears to be less sensitive for larger numbers. Example: d = 8 aligns to 1.655 and d = 9 aligns to 1.656 (third decimal place change).

Question: 5.

Navigate to page 3.1, another version of this continued fraction is included, this time without the fractional result and more sub-fractions. Use the slider to change the value of (d) and record the results for each value.

d	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
Decimal:	1.6154	1.6190	1.6206..	1.6216..	1.6222..	1.6226	1.6229	1.6232	1.6233	1.6235

What do you notice about the decimal values now that there are more 'sub-fractions'?

Answer: The decimal value is more 'stable'. Changing (d) only impacts the second decimal place and later the third decimal place.

Teacher Notes: The purpose of these two questions is to show students that with more repeated nested fractions, the decimal value approaches a specific value. This means we can at least estimate the value for an infinite continued fraction. This is the important distintion the activity aims to make, the difference between rational and irrations. This also helps students understand why 1/3 ≈ 0.3333 is still rational, even though it has infinitely many decimal places.

1) It can be expressed as the ratio of to numbers

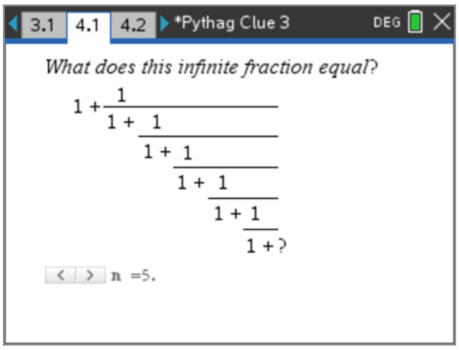
2) It can be (already is) expressed as a simple continued fraction, not an infinite continued fraction.

Author: P. Fox

Infinite Continued Fractions

Navigate to page 4.1. Use the slider to progressively reveal the fraction.

How can we calculate this fraction if it continues … for ever?



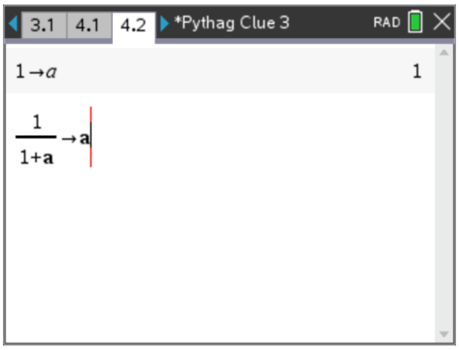
There is another way to approach this problem. Notice that for each section of the continued fraction, the (?) is replaced with the expression:

$$\frac{1}{1+?}$$

In mathematics we can use a variable to represent an unknown.

Navigate to page 4.2 (Calculator application). We need to seed '*a*' with a value. To store *a* as 1, press:

1 [ctrl] + [var] [A] [enter]



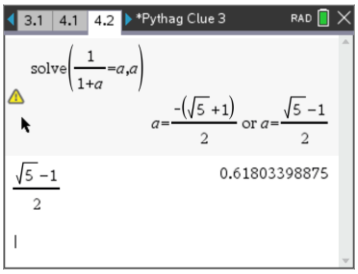
Now store the expression into *a*. [As shown opposite]

Question: 6.

Once the expression has been stored, keep pressing [enter] to see what happens. To calculate the approximate (≈) result press: [ctrl] + [enter].

What is the approximate value for the infinite fraction as it continues … on and on? [Fraction on page 4.1]
Note: You will need to add the original '1' from the very first line.

Answer: 1 + 0.61803398874985= 1.61803398874985 [This is φ, the golden ration!]



Teacher Notes: Advanced students may also like to try the 'solve' command:

Of course algebra such as this was not in existance at this historical time, mathematicians were simply trying to explore the notion of 'irrational numbers'.

The fact that even the square-roots of numbers could be written in terms of whole numbers, even though they go on indefinitely would have been much more palatable. The simple fraction 1/3 contains infinitely many decimal places.

Question: 7.

Navigate to Page 5.1, it contains a different 'infinite' fraction. Explore the approximate value for this infinite fraction.

Question: 8.

With the assistance of the following, determine a simple fraction that provides an excellent estimation for this infinite fraction.

Use the slider to progressively remove each sub-fraction, notice that a value has been inserted to replace the (?) used by the infinite fractions.

Author: P. Fox

Step 5: $1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{3}}}}}}$ $\therefore ? = \frac{1}{3}$,

Step 4: ? = $\frac{3}{7}$

Step 3: ? = $\frac{7}{17}$

Step 2: ? = $\frac{17}{41}$

Step 1: ? = $\frac{41}{99}$

Express the final result as an improper fraction. $\frac{338}{239}$

Express the final result as a decimal. 1.4142259414226

Comment: Students may notice that this number resembles $\sqrt{2} \approx 1.4142$

Question: 9.

Once you have identified the irrational number represented by the previous infinite fraction. Try and determine an infinite fraction for a different square-root value.

Explore

An interesting infinite fraction is of the form: $\cfrac{4}{1+\cfrac{1^2}{2+\cfrac{3^2}{2+\cfrac{5^2}{2+\cfrac{7^2}{2+\cfrac{9^2}{2+\ldots}}}}}}$

This infinite fraction has been set up in a program. Navigate to page 6.1 to see this fraction, then to page 6.2.

A program has been set up to start at the bottom of this fraction and work up … you can observe each row as it is computed. To run the program type: secrete() and press enter . You will be prompted for the number of iterations (sub fractions) you want to execute, up to 100. What number is being generated by this infinite fraction?

Author: P. Fox