Functions and Inverses

Student Activity

7 8 9 10 **11** 12

TI-Nspire™ Activity Student 4 hours

Watch the Video Tutorial

Introduction

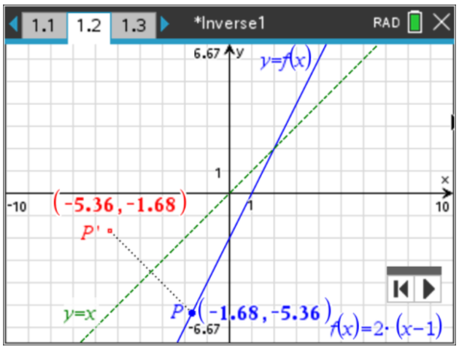
Part 1 of this activity involves using graphical and algebraic functionalities of TI-Nspire to explore the concept of the inverse of a function and the conditions for a function to have an inverse function. A practical application of the inverse function is also considered.

Part 2 involves exploring interesting properties of the graphs of a variety of functions and their inverses, including the location and quantity of intersection points between a function and its inverse.

PART 1. Understanding the inverse of a function

Please refer to the TI-Nspire document 'Inverse1'

Reflecting points in the line $y = x$



Open the TI-Nspire document 'Inverse1'.

Navigate to Page 1.2. P is a point on the graph of $y = f(x)$, where $f(x) = 2(x-1),\ x \in R$.

Point P' is the reflection of P in the line $y = x$.

P can be animated using the control buttons.
Alternatively, P can be grabbed (ctrl + ☒) and dragged along the line. Press esc to release the 'grabbed' P.

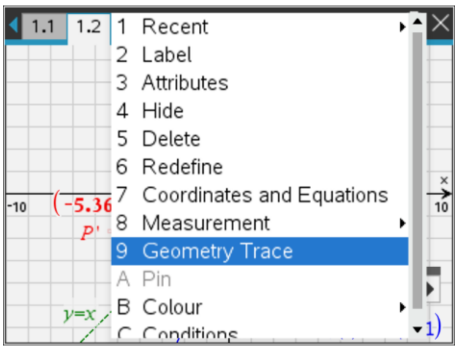
Question: 1.

Animate the point P and observe the coordinates of P and P' simultaneously.

a. What is the relationship between the two sets of coordinates?

b. Why does reflecting P in the line $y = x$ have this effect on the coordinates of P'?

Path of reflected points



Return P to the original position by clicking the left control button. The 'Geometry Trace' tool will be used to obtain an outline of the path of P'. To activate this tool, move the cursor to the point P' and open the context menu by pressing ctrl + menu.
Select 'Geometry Trace', then animate P. A trace of P' will be visible. When P' is outside the window settings, return P to the starting position, then exit the tool by pressing esc.

Question: 2.

Describe qualitatively some of the key features of the trace of P'. How do these key features relate to the graph of $y = f(x)$?

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Question: 3.

a. Use the trace of P' to calculate the following aspects of the line that contains all points on the trace. (For convenience, this line will be referred to as the 'reflected' line.)

  i. Axes intercepts

  ii. Gradient

b. Hence state the equation of the reflected line.

c. Test whether your equation in part (b). is correct by graphing your equation on page 1.2, using the same set of axes as the graph of $y = f(x)$. The line should contain all points on the trace.

Question: 4.

a. What is the relationship between the gradient of the reflected line and the gradient of the graph of $y = f(x)$?

b. What is the relationship between the axes intercepts of the reflected line and the graph of $y = f(x)$?

c. How could the equation of the reflected line be obtained from the equation of the original graph, $y = 2(x-1)$?

Technology Tip!

Erasing the 'Geometry Trace': move the cursor to a blank part of the graphing window and open the context menu ([ctrl]+[menu]). Select 'Erase Geometry Trace'.

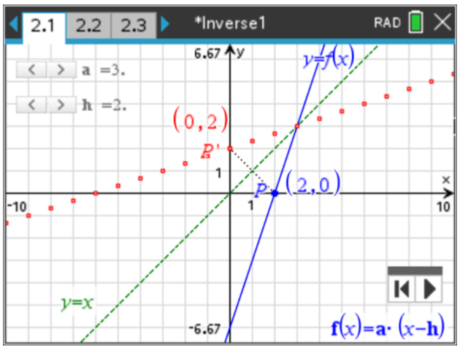
General case $y = a(x-h)$

Page 2.1 of the TI-Nspire document 'Inverse1' demonstrates the more general case where graphs with equations of the form $y = a(x-h)$ are reflected in the line $y = x$. Use the sliders to explore the effect of varying a and h on the equation of the reflected line.

Question: 5.

In the table below, state the equation of the reflected line. (i.e. Original line reflected in $y = x$)
Test whether your equation is correct by graphing your equation on page 2.1, using the same set of axes as the graph of $y = f(x)$. The line should contain all points on the trace.

Original equation	Equation of reflected line
a. $y = 3(x-2)$	
b. $y = \frac{1}{2}(x-4)$	
c. $y = 2(x+3)$	
d. $y = -2(x-3)$	
e. $y = -(x+4)$	



Inverse of a function

You would have observed that reflecting the graph of a function in $y = x$ results in the coordinates of every point on the graph being 'switched', so that $P(x, y) \to P'(y, x)$.

Therefore, the image of the reflection of $y = f(x)$ in $y = x$ has equation $x = f(y)$, which is said to be the equation of the **inverse** of the function f.

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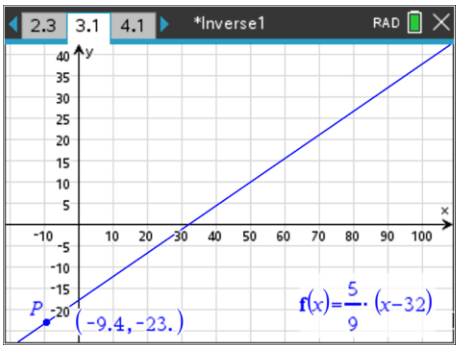
You would also have observed that if f is a linear function, then its inverse is also a linear function. The equation of the inverse can be obtained by 'switching' x and y in the original equation, then rearranging to make y the subject (i.e. solving for y).

An application of the inverse of a function: Historical temperature records

Australian temperature records prior to 1972 were recorded in Fahrenheit. To make comparisons with contemporary records, it is necessary to convert between Fahrenheit (F) and Celsius (C).

Page 3.1 of the TI-Nspire document 'Inverse1' shows the graph of $y = f(x) = \frac{5}{9}(x-32)$, where $y^\circ\mathrm{C}$ is the Celsius temperature at $x^\circ\mathrm{F}$ (Fahrenheit). P is a point on the graph of f.



Question: 6.

a. Find the coordinates of the axes intercepts and explain what information about the temperature scales is given by these values.

  i. x-intercept

  ii. y-intercept

b. Explain what information about the temperature scales is given by the gradient of the graph.

Question: 7.

Use a graphical method to convert the following temperatures from Fahrenheit to Celsius, correct to one decimal place. Confirm the answers by substituting (for x) each value in the expression $f(x)$. Do this in the Calculator application on page 3.2.

a. $-15^\circ\mathrm{F}$ b. $20^\circ\mathrm{F}$ c. $67^\circ\mathrm{F}$ d. $100^\circ\mathrm{F}$

Inverse of function f: Celsius to Fahrenheit

Fahrenheit is still the official temperature scale in the USA. For the benefit of friends in the USA, you may sometimes want to state temperatures in both Celsius and Fahrenheit (for example, when posting on social media).

Question: 8.

Suppose that you wanted to post about a temperature of $28^\circ\mathrm{C}$ in both temperature scales.

a. Using the graph on page 3.1, convert $28^\circ\mathrm{C}$ to Fahrenheit, correct to one decimal place.

b. Confirm the answer using an algebraic method in the Calculator application on page 3.3.

We saw in Question 7 that the function f tells us the value of y if we know the value of x.
The **inverse** of the function f will tell us what the value x had to be to get that value of y.

Notation for inverse of a function

You have already seen that the inverse of a *linear* function, f, is itself a linear function.

The inverse function is denoted by f^{-1}.

Example: if $f: R \to R,\ f(x) = 4(x-5)$ then $f^{-1}: R \to R,\ f^{-1}(x) = \frac{x}{4} + 5$

Caution! f^{-1} means 'f *inverse*'. It does NOT mean the reciprocal of f.

Rule of f^{-1} if $f(x) = \frac{5}{9}(x-32)$

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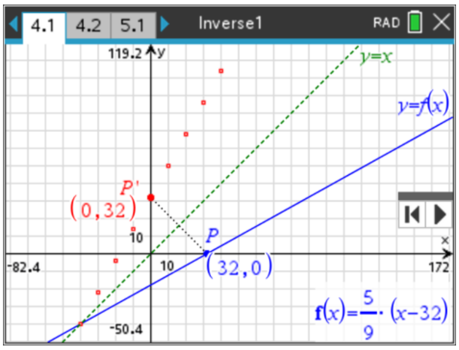
Page 4.1 shows the graph of $y = f(x) = \frac{5}{9}(x - 32)$ with different window settings, and a trace of the image of P reflected in the line $y = x$.



Question: 9.

a. Find the rule of f^{-1}, the inverse of f.

b. Test whether your equation $y = f^{-1}(x)$ is correct by graphing your equation on page 4.1. The graph should contain all points on the trace.

Question: 10.

Use the graph of $y = f^{-1}(x)$ to convert the following temperatures from Celsius to Fahrenheit.

a. $-25^{\circ}\mathrm{C}$ b. $10^{\circ}\mathrm{C}$ c. $53^{\circ}\mathrm{C}$

Question: 11.

a. At what temperature do Celsius to Fahrenheit have the same value?

b. Explain how you can determine this temperature from the graphs on page 4.1.

Question: 12.

- The lowest possible theoretical temperature, known as absolute zero, is equivalent to $-273.15^{\circ}\mathrm{C}$ by international agreement.
- Assume that the highest possible temperature (known as the Planck temperature) is so high that it is modelled as infinite.

a. Use the above information to determine the domain and range of f^{-1} and f.

b. Comment on any interesting aspect(s) of the domain and range of f and f^{-1}.

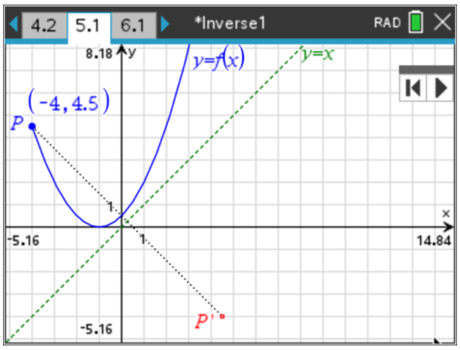
Inverse of some non-linear functions

Page 5.1 of the TI-Nspire document 'Inverse1' shows the graph of

$$f:[-4,5] \rightarrow R,\ f(x) = \frac{1}{2}(x+1)^2.$$

The point P' is the reflection of P in the line $y = x$.

P can be animated using the control buttons ⏮ ▶.
Alternatively, P can be grabbed (ctrl + ☒) and dragged along the line. Press esc to release the 'grabbed' P.



Question: 13.

Describe qualitatively some of the key features of the trace of P'. How do these key features relate to the graph of $y = f(x)$?

Question: 14.

Consider the curve that contains all points of the trace of P'. Could this curve be the graph of a *functional* relation (i.e. the graph of a relation that is also a function)? Explain the reason(s) for your answer.

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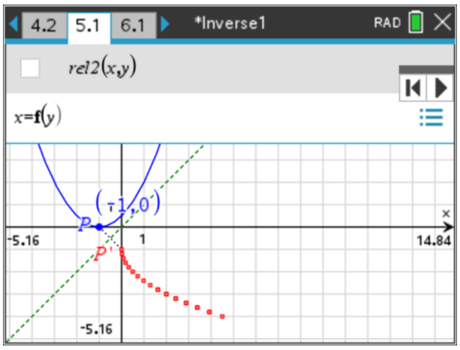
Relation Graphing

Page 5.1 should show the graph $y = f(x)$ and a trace of the inverse of f. The graph of the inverse of f is a reflection of $y = f(x)$ in the line $y = x$. This results in the 'switching' of the x and y coordinates of every point on the graph of the inverse. Therefore, the equation of the inverse graph could be expressed as $x = f(y)$.



Steps to graph the relation $x = f(y)$ on page 5.1 menu > Graph Entry/Edit > Relation. Input $x = f(y)$ in the entry line and press enter.

Question: 15.

a. What is the relationship between the relation $x = f(y)$ and the trace of P'?

b. In the Calculator application page 5.2, solve the equation $x = f(y)$ for y (that is, $\text{solve}(x = f(y), y)$).

Write down the answer. Explain what the answer means, and how it relates to the graph of $x = f(y)$.

One-to-one functions

Question: 16.

If $f : [-4, 5] \to R,\ f(x) = \frac{1}{2}(x+1)^2$, find the values of x such that $f(x) = \frac{5}{2}$. By referring to the graph of f, explain why there are two possible values of x.

What is a one-to-one function?

If no horizontal line intersects the graph of the function more than once, then the function is said to be one-to-one. Therefore:

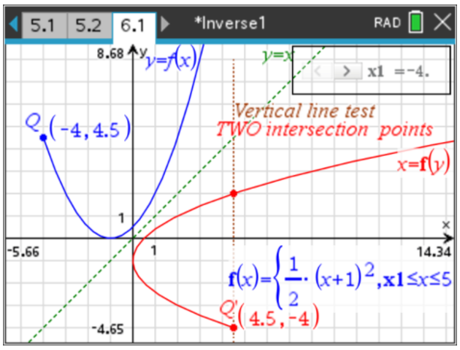
- all the linear functions considered in Questions 1 to 12 are one-to-one functions, and their inverses are also functions;
- $f : [-4, 5] \to R,\ f(x) = \frac{1}{2}(x+1)^2$ is an example of a function that is **not** one-to-one, and its inverse relation is **not** a function.

Restricting the domain to obtain a one-to-one function

Page 6.1 shows the graph of the function

$f : [x1, 5] \to R,\ f(x) = \frac{1}{2}(x+1)^2$, where $-4 \le x1 < 5$,

and the graph of the inverse of f.



Use the slider to adjust the value of $x1$. This will dynamically adjust the domain of the graph of f, and the range of the graph of the inverse of f.

Question: 17.

a. What is the smallest value of $x1$ for which f is a one-to-one function?

b. When f is a one-to-one function, what effect does this have on the inverse of f?

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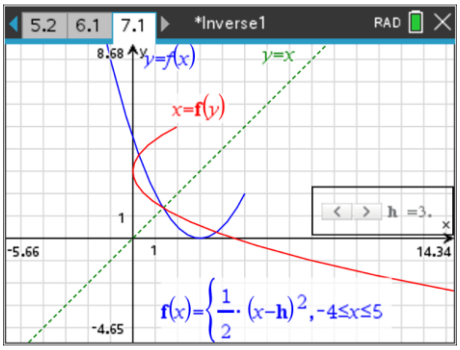
Exploring a family of functions

Now consider the family of functions of the form

$$f:[-4,5]\rightarrow R,\ f(x)=\frac{1}{2}(x-h)^2,\ \text{where } -5\le h\le 10$$

Page 7.1 shows the graphs of the function and its inverse.

Use the slider to adjust the value of h.



Question: 18.

a. For what values of h is f a one-to-one function?

b. Comment on any interesting aspects of the graphs of f and its inverse, for various values of h.

PART 2

Question: 19.

Consider the function $f:[-2,\infty)\rightarrow R$, $f(x)=2x-1$. Find the inverse function, $f^{-1}(x)$ stating the domain and range. Find the coordinates of the points where the graphs of the function and the inverse cross the coordinate axes. Solve the equations $f(x)=x$, $f^{-1}(x)=x$, $f(x)=f^{-1}(x)$.

Sketch the graphs of $y=f(x)$, $y=f^{-1}(x)$ and $y=x$. Comment on your observations.

Question: 20.

Consider the function $f:R\rightarrow R$, $f(x)=x$. Find the inverse function, $f^{-1}(x)$ stating the domain and range. Solve the equations $f(x)=x$, $f^{-1}(x)=x$, $f(x)=f^{-1}(x)$.

Comment on your observations.

Question: 21.

Consider the function $f:R\rightarrow R$, $f(x)=x+2$. Find the inverse function, $f^{-1}(x)$ stating the domain and range. Find the coordinates of the points where the graphs of the function and the inverse cross the coordinate axes. Solve the equations $f(x)=x$, $f^{-1}(x)=x$, $f(x)=f^{-1}(x)$.

Sketch the graphs of $y=f(x)$, $y=f^{-1}(x)$ and $y=x$. Comment on your observations.

Question: 22.

Consider the function $f:R\backslash\{0\}\rightarrow R$, $f(x)=\frac{1}{x}$. Find the inverse function, $f^{-1}(x)$ stating the domain and range and equations of any asymptotes. Solve the equations

$f(x)=x$, $f^{-1}(x)=x$, $f(x)=f^{-1}(x)$.

Sketch the graphs of $y=f(x)$, $y=f^{-1}(x)$ and $y=x$. Comment on your observations.

Question: 23.

Consider the function $f:R\backslash\{0\}\rightarrow R$, $f(x)=-\frac{1}{x}$. Find the inverse function, $f^{-1}(x)$ stating the domain and range and equations of any asymptotes. Solve the equations

$f(x)=x$, $f^{-1}(x)=x$, $f(x)=f^{-1}(x)$.

Sketch the graphs of $y=f(x)$, $y=f^{-1}(x)$ and $y=x$. Comment on your observations.

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Question: 24.

Consider the function $f:R\to R$, $f(x)=-x^3$. Find the inverse function, $f^{-1}(x)$ stating the domain and range. Solve the equations $f(x)=x$, $f^{-1}(x)=x$, $f(x)=f^{-1}(x)$. Sketch the graphs of $y=f(x)$, $y=f^{-1}(x)$ and $y=x$. Comment on your observations.

Question: 25.

Consider the function with the rule $f(x)=\dfrac{2}{x-4}+3$. Find the inverse function, $f^{-1}(x)$ stating the domain and range and equations of any asymptotes. Sketch the graphs of $y=f(x)$, $y=f^{-1}(x)$ and $y=x$. Comment on your observations.

Question: 26.

Consider the function $f:R\setminus\{0\}\to R$, $f(x)=-\dfrac{1}{x^3}$. Find the inverse function, $f^{-1}(x)$ stating the domain and range and equations of any asymptotes. Solve the equations $f(x)=x$, $f^{-1}(x)=x$, $f(x)=f^{-1}(x)$.
Sketch the graphs of $y=f(x)$, $y=f^{-1}(x)$ and $y=x$. Comment on your observations.

Question: 27.

Consider the function $f:R\setminus\{1\}\to R$, $f(x)=\dfrac{x-a}{x-1}$, where $a\neq 1$. Find the inverse function, $f^{-1}(x)$ stating the domain and range and equations of any asymptotes. Comment on your observations.

Question: 28.

Consider the function $f:D\to R$, $f(x)=\sqrt{9-3x}$, where *D* is the maximal domain for which *f* is defined. Find the inverse function, $f^{-1}(x)$ stating the domain and range. Solve the equations $f(x)=x$, $f^{-1}(x)=x$, $f(x)=f^{-1}(x)$.
Sketch the graphs of $y=f(x)$, $y=f^{-1}(x)$ and $y=x$. Comment on your observations.
Find other functions of the form $g:D\to R$, $g(x)=\sqrt{b-ax}$ which have similar properties?

Question: 29.

a) Consider the function $f:R\to R$, $f(x)=x^2-6x+8$. Find the inverse relation and find the coordinates of the points of intersection between the function and its inverse relation. Sketch the graph of the function *f* and the inverse relation.

b) Let $g_1:D_1\to R$, $g_1(x)=x^2-6x+8$. Find D_1 the largest subset of *R*, such that g_1 is a one-to-one increasing function, hence find the inverse function and the points of intersection between the function and its inverse.

c) Let $g_2:D_2\to R$, $g_2(x)=x^2-6x+8$. Find D_2 the largest subset of *R*, such that g_2 is a one-to-one decreasing function, hence find the inverse function and the points of intersection between the function and its inverse.

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Question: 30.

Several students joined the conversation and stated some comments regarding functions, their inverses and points of intersection between functions and their inverses.

- Ashley stated the reciprocal and inverse of a function are the same.
- Belinda stated that if a linear function has a gradient of *m*, then a line perpendicular has a gradient of $-\frac{1}{m}$ and the inverse function has a gradient of $\frac{1}{m}$.
- Colin stated that when we sketch graphs of a function and its inverse, the two graphs are reflections in the line $y = x$.
- Daisy stated that the domain of the function *f* is equal to the range of the inverse function and that the domain of the inverse function is equal to the range of the original function.
- Edward stated that if the graph of a function crosses the *x*-axis at the point (a,b) then the graph of the inverse function crosses the *y*-axis at the point (b,a).
- Frank stated that if a function has the line $x = a$ as a vertical asymptote, then the inverse function has the line $y = a$ as a horizontal asymptote, furthermore if a function has the line $y = b$ as a horizontal asymptote, then the inverse function has the line $x = b$ as a vertical asymptote.
- George stated that the graphs of a function and its inverse must intersect only on the line $y = x$.
- Henriette stated that the graphs of a function and its inverse may not intersect.
- Ian stated that the graphs of a function and it inverse may intersect once on the line $y = x$.
- Jack stated that the graphs of a function and its inverse may intersect an infinite number of times.
- Kevin stated that the graphs of a function and its inverse may intersect at two distinct points, both of which lie on the $y = x$.
- Lilly stated that it is possible for the graphs of a function and its inverse to intersect once on the line $y = x$ and again at two other points distinct points $(a,b),(b,a)$ where $a \neq b$.
- Myer stated that it is possible for the graphs of a function and its inverse to intersect at two distinct points (a,b) ,(b,a) where $a \neq b$, both of which do not lie on the lie $y = x$.
- Nancy stated that it is possible for the graphs of a function and its inverse relation to intersect at four points.
- Oliver stated that it is possible for the graphs of a function and its inverse function to intersect at four points.
- Peter stated that the solution of the equations $f(x) = x$, $f^{-1}(x) = x$, $f(x) = f^{-1}(x)$ always all give the same solutions.

Based on the previous questions, comment on the validity of all these student's statements.

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