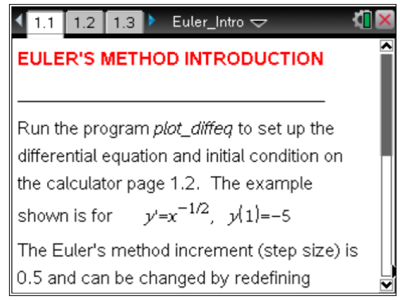


Math Objectives

- Students will use Euler's method to create a graphical approximation to the solution to a differential equation.
- Students will use Euler's method to find an approximate numerical function value of the solution to a differential equation.
- Students will describe how various factors affect the accuracy of Euler's method, including initial condition and step size.
- Students will look for and make use of structure. (CCSS Mathematical Practice)
- Students will reason abstractly and quantitatively. (CCSS Mathematical Practice)

Vocabulary

- Euler's method
- differential equation
- initial condition
- step size
- local linearity
- concavity

About the Lesson

- This lesson involves using Euler's method to visualize the graph of an approximate solution to a differential equation and to estimate a specific value of a solution
- As a result, students will:
  - Learn about Euler's method graphically and analytically.
  - Conjecture about factors that affect the accuracy of this method.
  - Visualize and make sense out of an important numerical technique in the study of elementary differential equations.

TI-Nspire™ Navigator™ System

- Use Screen Capture to demonstrate various approximations depending on the initial condition and step size.
- Use Teacher Edition computer software to review student work.
- Encourage students to try other combinations of initial condition and step size and to present their screen.

TI-Nspire™ Technology Skills:

- Download a TI-Nspire document
- Open a document
- Move between pages

Tech Tips:

- Make sure the font size on your TI-Nspire handheld is set to Medium.
- In Graphs, you can view the function entry line by pressing ctrl G, and then enter a function.
- The arguments for the calculator function **deSolve** are (differential equation, independent variable, dependent variable).

Lesson Materials:

Student Activity
Euler_Intro_Student.pdf
Euler_Intro_Student.doc

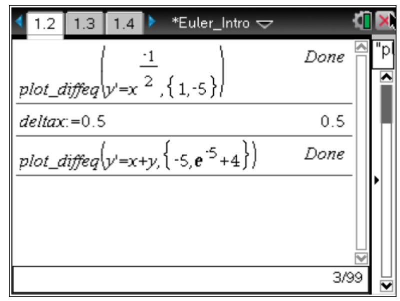
TI-Nspire document
Euler_Intro.tns

Visit www.mathnspired.com for lesson updates and tech tip videos.

Discussion Points and Possible Answers

Tech Tip: If students experience difficulty executing and visualizing a step in Euler's method, check to make sure that they have moved the cursor to the left side of page 1.4. Tap the up arrow to compute and plot the next Euler's method point.

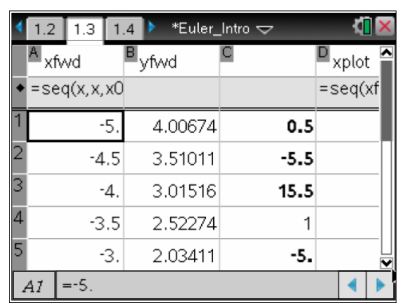
Move to page 1.2.



Consider the differential equation $y' = x + y$. You will use Euler's method to find several approximations for the value of *y*, the solution to this differential equation, at $x = 0$.

1. Suppose you know the graph of the solution to the differential equation passes through the point $x_0 = -5$, $y_0 = e^{-5} + 4 \approx 4.007$ (the initial condition). The default value for the step size is $\Delta x = 0.5$. Enter the following command on page 1.2: $plot\_diffeq\left(y' = x + y, \left\{-5, e^{-5} + 4\right\}\right)$.

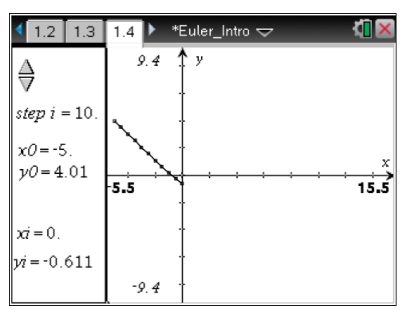
Move to page 1.3.



xfwd represents the value of *x* at each step in the approximation, and **yfwd** represents the approximate value of *y* at each step. Use the spreadsheet on page 1.3 to complete the following tables.

i	0	1	2	3	4	5
x_i	–5.0	–4.5	–4	–3.5	–3.0	–2.5
y_i	**4.007**	**3.510**	**3.015**	**2.523**	**2.034**	**1.551**

i	6	7	8	9	10
x_i	–2.0	–1.5	–1.0	–0.5	0
y_i	**1.077**	**0.615**	**0.173**	**–0.241**	**–0.611**

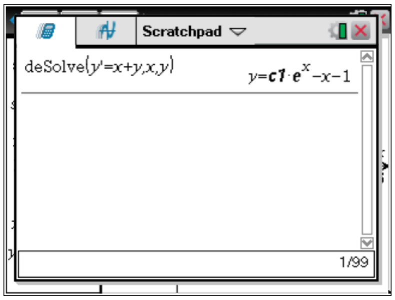


Move to page 1.4.

2. On page 1.4, increase *step i* 10 times by tapping the slider arrow to visualize Euler's method. Based on this graph and the table above, what is the approximate value of $y(0)$?

   Answer: The graph and the spreadsheet suggest $y(0) \approx -0.611$.

3. Open a Calculator Scratchpad and use the function **deSolve** to find the general solution to the differential equation. Verify analytically that the function $y = e^x - x - 1$ is a solution to the differential equation that satisfies the initial condition.



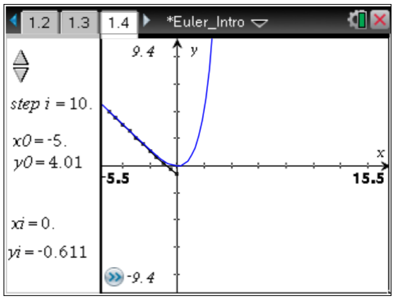
   Answer: The Calculator function indicates that the general solution to the differential equation is $y = ce^x - x - 1$ where *c* is a constant.

   Verification (student solutions may vary):

$$\begin{aligned} y' &= e^x - 1 \\ &= e^x - 1 + (x - x) \\ &= x + (e^x - x - 1) \\ &= x + y \end{aligned}$$

$$y(-5) = e^{-5} - (-5) - 1 = e^{-5} + 4$$

4. Find $y(0)$. Is the approximation in question 2 an overestimate or underestimate? Add the graph of the solution to the right side of page 1.4. Notice how Euler's method produces successive approximations very close to the true value of *y* until *x* gets close to 0. Use the characteristics of the graph of the solution to explain this observation.



   Answer: $y(0) = e^0 - 0 - 1 = 1 - 1 = 0$. The approximation from question 2, $y(0) = -0.611$, is an underestimate of the true value.

   The graph of the solution is approximately linear from $x = -5$ to $x = -2$. Since Euler's method is based on the slope of the tangent line, the approximations are very close to the true solution. From $x = -2$ to $x = 0$, the graph of the solution has a sharp curve and is concave up. Using the tangent line to the graph of the solution, the approximations are less than the true values.

5. The starting point and the step size in Euler's method may affect the approximation. On page 1.2, change the value of Δx and the initial condition (using the function *plot_diffeq*) as indicated and use page 1.3 or page 1.4 to complete the following table.

y_0	$e^{-2.5}+1.5$	$e^{-2.5}+1.5$	e^{-1}	e^{-1}	e^{-1}	$e^{-0.5}-0.5$
Δx	0.5	0.25	0.5	0.25	0.1	0.1
$y(0)\approx$	**–0.377**	**–0.236**	**–0.172**	**–0.102**	**–0.045**	**–0.023**

Which combination of initial condition and step size produces the best approximation? Why?

Answer: The initial condition $(x_0, y_0) = (-0.5, e^{-0.5} - 0.5)$ produces the best approximation. This choice uses an initial point close to (0, 0) and also uses a small step size.

6. Explain how each of the following affects an approximation using Euler's method.
   a. The initial condition, (x_0, y_0).

   Answer: Let x_a be the value of *x* for which an approximate value of the true solution is needed. The closer x_0 is to x_a, the better the approximation. Intuitively, there will be fewer steps in Euler's method to reach x_a. Therefore, there are fewer calculations that introduce error.

   b. The step size, Δx.

   Answer: The table in question 5 suggests the smaller the step size, the smaller the approximation error. Note that for fixed initial condition, a smaller step size will require more steps/calculations.

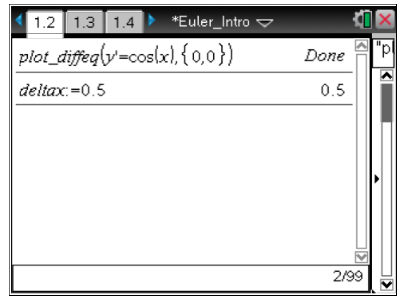
   c. The local linearity of the graph of the solution to the differential equation.

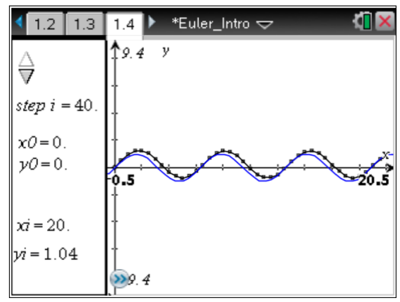
   Answer: The more linear the graph of the solution, the more accurate the approximation. Euler's method relies on the tangent line to the graph of the solution. If the graph of the solution is locally linear, then the graph of the tangent line will be very close to the solution.

   d. The concavity of the graph of the solution to the differential equation.

   Answer: If the graph of the solution is concave up, then Euler's method will produce an underestimate. If the graph of the solution is concave down, then Euler's method will produce an overestimate.

7. Enter the appropriate command on page 1.2 to consider the differential equation $y' = \cos x$ with initial condition (0, 0) and $\Delta x = 0.5$. On page 1.4, use the up arrow to visualize 40 points on the graph of the solution curve using Euler' method. What is the solution to this differential equation suggested by Euler's method? Add a graph of your guess to the right side of page 1.4. How close is the graph produced by Euler's method to the solution? Why?





Answer: The solution to this differential equation suggested by Euler's method is $y = \sin x$. The graph produced by Euler's method is very close to the graph of $y = \sin x$.

Teacher Tip: Have students verify analytically that $y = \sin x$ is the solution to this differential equation. Ask students why, in this case, Euler's method produces such a good approximation.

Wrap Up

Upon completion of this discussion, the teacher should ensure that students understand:

- The relationship between the initial condition, step size, linearity, and concavity and the approximation produced using Euler's method.
- How Euler's method can be used to produce the graph of an approximate solution to a differential equation.

At the end of this activity, you might consider having students discover that Euler's method can provide a visual and/or numerical approximation to a solution even if there is no symbolic solution (when the calculator function **deSolve** fails). For example, consider the differential equation $y' = x\sin(x)\ln(x)$ with initial condition (1, 0). Ask students to find a solution analytically, to find a solution using the calculator, and to use Euler's method to sketch a graph of the approximate solution.