Foci Definition of Ellipses and Hyperbolas

Math Objectives

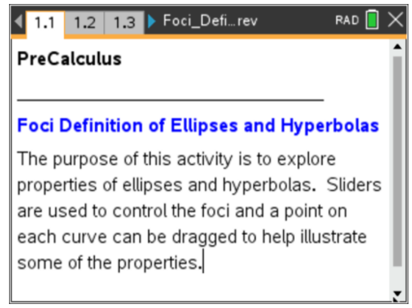
- Students will be able to define an ellipse as the set of points whose distances to two fixed points (foci) have a constant sum.
- Students will be able to define a hyperbola as the set of points whose distances to two fixed points (foci) have a constant difference.
- Students will be able to describe the relationship between the location of the foci and the shapes of the corresponding ellipses and hyperbolas.
- Students will be able to determine the effect of the eccentricity of ellipses and hyperbolas on the shape of their curves.
- Students will reason abstractly and quantitatively (CCSS Mathematical Practice).
- Students will look for and make use of structure (CCSS Mathematical Practice).
- Students will use technological tools to explore and deepen understanding of concepts (CCSS Mathematical Practice).

Vocabulary

- conjugate axis
- eccentricity
- ellipse
- focus/foci
- hyperbola
- major axis
- minor axis
- semi-major axis
- semi-minor axis
- transverse axis
- vertex of a conic

About the Lesson

- This lesson involves observing and describing the relationships between the foci of ellipses and hyperbolas and the shape of the corresponding curves.
- As a result, students will:
  - Define an ellipse as the set of points whose distances to two fixed points (foci) have a constant sum.
  - Define a hyperbola as the set of points whose distances to two fixed points (foci) have a constant difference.
  - Manipulate sliders to observe the relationship between the foci and sum/difference of the distances from the foci to a point on the curve.
  - Observe the effect of the relationship between the foci and the shapes of ellipses or hyperbolas.



TI-Nspire™ Technology Skills:

- Download a TI-Nspire document
- Open a document
- Move between pages
- Grab and drag a point

Tech Tips:

- Make sure the font size on your TI-Nspire handhelds is set to Medium.
- You can hide the function entry line by pressing ctrl G.

Lesson Files:

Student Activity
Foci_Definition_of_Ellipses_and_Hyperbolas_Student.pdf
Foci_Definition_of_Ellipses_and_Hyperbolas_Student.doc

TI-Nspire document
Foci_Definition_of_Ellipses_and_Hyperbolas.tns

Visit www.mathnspired.com for lesson updates and tech tip videos.

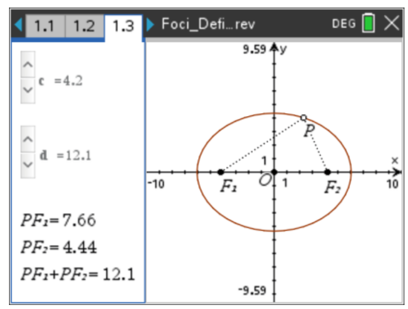
TI-Nspire™ Navigator™ System

- Transfer a File.
- Use Live Presenter to provide assistance to students throughout the activity.
- Use Screen Capture to monitor students' progress.
- Use Quick Poll to assess students' understanding.

Discussion Points and Possible Answers

Teacher Tip: Although this activity covers the foci definition of both ellipses and hyperbolas, you might want to break it up into two separate lessons, covering ellipses and Problem 1 on one day and hyperbolas and Problem 2 on another.

Move to page 1.3.



1. Points F_1 and F_2 are called the foci (plural of focus) of the ellipse pictured. Point O is the center of the ellipse, and point P lies on the ellipse.

Teacher Tip: Be aware that the values of PF_1 and PF_2 are rounded, so they might not agree with values returned by the measurement tool.

a. Use the sliders to select arbitrary values for c and d. Set the constant sum with the slider d. Set the foci $(\pm c, 0)$ with the slider c. Click on point *P,* and drag it around the ellipse. Describe what happens to:
   - points F_1 and F_2.
   - distances PF_1 and PF_2.
   - the sum $PF_1 + PF_2$.

<u>**Sample Answers:**</u> As point P is dragged around the ellipse, points F_1 and F_2 remain fixed and the sum $PF_1 + PF_2$ remains constant while the distances PF_1 and PF_2 change. As PF_1 gets larger, PF_2 gets smaller, and conversely.

TI-Nspire Navigator Opportunity: ***Screen Capture and Live Presenter***
See Note 1 at the end of this lesson.

b. Use the slider to select an arbitrary value for d. Click on the slider arrows that correspond to c, and describe what happens to:
   - the graph of the ellipse.
   - points F_1 and F_2.
   - distances PF_1 and PF_2.
   - the sum $PF_1 + PF_2$.

   Sample Answers: As the value of *c* increases, the graph of the ellipse compresses vertically and remains the same horizontally. As the value of *c* decreases, the graph of the ellipse stretches vertically and remains the same horizontally. Points F_1 and F_2 as well as the distances PF_1 and PF_2 change as the value of *c* changes. However, the sum $PF_1 + PF_2$ remains constant.

c. As you click the slider, observe what is happening to points F_1 and F_2 on the graph. What is the relationship between the variable *c* and points F_1 and F_2?

   Sample Answers: The variable *c* determines the distance between the center of the ellipse and each of its foci.

d. Use the slider to select an arbitrary value for c. Click on the slider arrows that correspond to d and describe what happens to:
   - the graph of the ellipse.
   - points F_1 and F_2.
   - distances PF_1 and PF_2.
   - the sum $PF_1 + PF_2$.

   Sample Answers: As the value of *d* increases, the graph of the ellipse stretches both vertically and horizontally. As the value of *d* decreases, the graph of the ellipse compresses both vertically and horizontally. Points F_1 and F_2 remain fixed. However the distances PF_1 and PF_2 as well as the sum $PF_1 + PF_2$ change as the value of *d* changes.

e. Determine the relationship between the variable *d* and distances PF_1 and PF_2.

   Sample Answers: The variable *d* is the constant sum $PF_1 + PF_2$.

Teacher Tip: You might want to review some of these questions with the entire class to ensure that all students understand how changes in c and d affect the foci, the distances, and the sum.

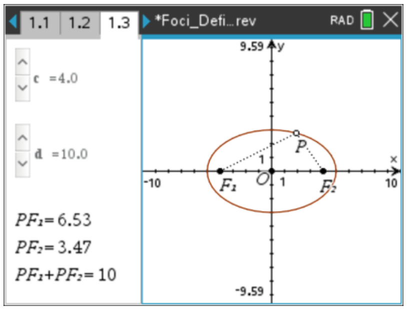
Be aware that there is no protection against changing values of *c* and/or *d* that will cause the graph to disappear. This might make a good point for class discussion.

2. Based on your observations in Question 1, fill in the blanks in the following definition of an ellipse:

Answer: An ellipse is the set of points $P(x, y)$ in a plane such that the **sum** of the distances from two fixed points F_1 and F_2, called the foci, is **constant**.

TI-Nspire Navigator Opportunity: ***Quick Poll***
See Note 2 at the end of this lesson.

3. Click the sliders to set *c* = 4 and *d* = 10. Click and drag point *P* so that it lies on the *x*-axis at $(5,0)$ with $PF_1 = 9$ and $PF_2 = 1$.



Tech Tip: You can display the coordinates of point *P* by clicking on the graphing window to activate it. Select **MENU > Actions > Coordinates and Equations**. Click on point *P*, and press enter enter.

a. What is the length of $\overline{OP}$, the **semi-major axis**?

Sample Answers: Since the coordinates of *O* are $(0,0)$ and the coordinates of *P* are $(5,0)$, $OP = 5$.

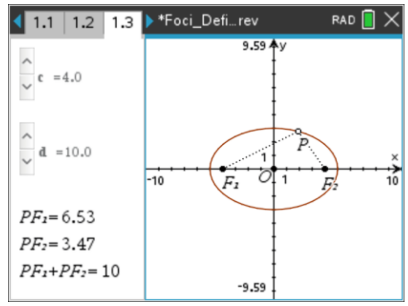
b. What relationship exists between OP, PF_1, and PF_2?

Sample Answers: Since $OP = 5$, $PF_1 = 9$ and $PF_2 = 1$, $OP = \frac{1}{2}(PF_1 + PF_2)$.

c. What is the relationship between the length of the major axis and the sum of the distances from the foci to a point on the ellipse?

<u>Sample Answers:</u> The length of the major axis is equal to the sum of the distances from the foci to a point on the ellipse.

4. Set *c* = 4 and *d* = 10. Click and drag point *P* so that it lies on the y-axis at $(0,3)$ and $PF_1 = PF_2 = 5.$



a. What are the lengths of OP, the **semi-minor axis**, and OF_1?

<u>Sample Answers:</u> Since the coordinates of *O* are $(0,0)$ and the coordinates of *P* are $(0,3)$, $OP = 3.$ Since the coordinates of *O* are $(0,0)$ and the coordinates of F_1 are $(4,0)$, $OF_1 = 4.$

b. What relationship exists among OP, OF_1, and PF_1? Among OP, OF_2, and PF_2?

<u>Sample Answers:</u> Since PF_1 and PF_2 are hypotenuses of right triangles OPF_1 and OPF_2, respectively, we know that $(OP)^2 + (OF_1)^2 = (PF_1)^2$ and $(OP)^2 + (OF_2)^2 = (PF_2)^2$. We can phrase this verbally as, the square of the length of the semi-minor axis plus the square of the distance from the center of the ellipse to the focus equals the square of the length of the semi-major axis. We can verify this by substituting the values $OP = 3$, $OF_1 = 4$, and $PF_1 = 5$ into the equation $(OP)^2 + (OF_1)^2 = (PF_1)^2$.

5. The shape of an ellipse is determined by its eccentricity, a number that indicates how elongated a conic section is. The eccentricity, e, of a horizontal ellipse is defined as $e = \frac{c}{a}$, where *c* is the distance from the center of the ellipse to a focus, and *a* is the horizontal distance from the center to the vertex.

Teacher Tip: You might want to remind students that e, the eccentricity of a conic section, has no relationship to the irrational number, e.

a. Continue with d = 10. As you click on the *c*-slider, describe what happens to the shape of the ellipse as *c* gets larger and as *c* gets smaller and why.

Sample Answers: As *c* gets larger, the graph of the ellipse becomes less circular and more oval. As *c* gets smaller, the graph of the ellipse gets more circular and less oval. Since *c* determines the distance between the center of the ellipse and each of its foci, as *c* gets larger, the foci move farther away. As *c* gets smaller, the foci move closer together—so close that they eventually converge to a single point. As the two foci get closer together, PF_1 and PF_2 also get closer together.

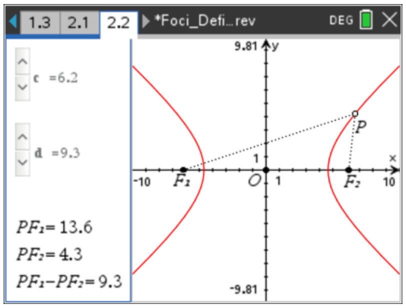
b. What is the shape of the ellipse if *c* = 0? Explain your answer.

Sample Answers: The ellipse becomes a circle since when *c* = 0, the foci and the center are the same point, so that $PF_1 = PF_2$.

c. Using the information above, give the range of values for *e*, the eccentricity of an ellipse.

Sample Answers: From part (b) above, if *c* = 0, the ellipse becomes a circle. From part (a) above, as *c* gets closer to *a*, the graph of the ellipse gets flatter and flatter. If *c* = *a*, the foci and vertices would be the same, and there would be no ellipse. Thus, $0 < e < 1$. Algebraically, since $0 < c < a$, divide all terms by *a* to produce $0 < \frac{c}{a} < 1$. Since $e = \frac{c}{a}$, we see that $0 < e < 1$.

Move to page 2.2.



6. Points F_1 and F_2 are called the foci of the hyperbola pictured. Point O is the center of the hyperbola, and point P lies on the hyperbola.

Teacher Tip: Be aware that the values of PF_1 and PF_2 are rounded, so they may not agree with values returned by the measurement tool.

a. Use the sliders to select an arbitrary value for both c and d. Set the constant difference with the slider d. Set the foci $(\pm c, 0)$ with the slider c. Click on point *P,* and drag it around the graph of the hyperbola. Describe what happens to:

- points F_1 and F_2.
- distances PF_1 and PF_2.
- the difference $PF_1 - PF_2$.

Sample Answers: As point P is dragged around the graph of the hyperbola, points F_1 and F_2 remain fixed, and the difference $PF_1 - PF_2$ remains constant. However, the distances PF_1 and PF_2 change. As PF_1 gets larger, PF_2 also gets larger and, as PF_1 gets smaller, PF_2 also gets smaller, but the difference, $PF_1 - PF_2$, is constant.

b. Use the slider to select an arbitrary value for d. Click on the *c*-slider, and describe what happens to:

- the graph of the hyperbola
- points F_1 and F_2.
- distances PF_1 and PF_2.
- the difference $PF_1 - PF_2$.

Sample Answers: As the value of *c* increases, the hyperbola opens wider. As the value of *c* decreases, the hyperbola compresses. Points F_1 and F_2 as well as the distances PF_1 and PF_2 change as the value of *c* changes. However, the difference $PF_1 - PF_2$ remains constant.

c. As you click the slider, observe what is happening to points F_1 and F_2 on the graph. What is the relationship between the variable *c* and points F_1 and F_2?

Sample Answers: The variable *c* determines the distance between the center of the hyperbola and each of its foci.

d. Use the slider to select an arbitrary value for c. Click on the *d*-slider and describe what happens to:

- the graph of the hyperbola.
- points F_1 and F_2.
- distances PF_1 and PF_2.
- the difference $PF_1 - PF_2$.

Sample Answers: As the value of *d* increases, the hyperbola compresses and the two branches of the hyperbola move further apart. As the value of *d* decreases, the hyperbola stretches and the two branches of the hyperbola move closer together. Points F_1 and F_2 remain fixed. However the distances PF_1 and PF_2 as well as the difference $PF_1 - PF_2$ change as the value of *d* changes.

e. Determine the relationship between the variable *d* and distances PF_1 and PF_2.

Sample Answers: The variable *d* is the constant difference $PF_1 - PF_2$.

TI-Nspire Navigator Opportunity: *Screen Capture*
See Note 3 at the end of this lesson.

Teacher Tip: Be aware that there is no protection against changing values of *c* and/or *d* that will cause the graph to disappear. This might make a good point for class discussion.

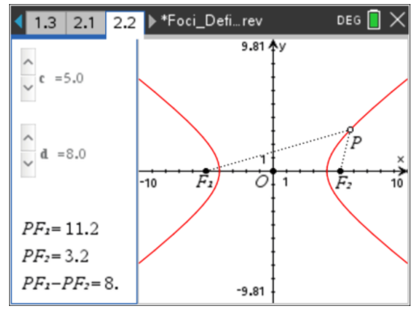
7. Based on your observations in Question 6, fill in the blanks in the following definition of a hyperbola.

Answer: A hyperbola is the set of points $P(x, y)$ in a plane such that the absolute value of **the difference** of the distances from two fixed points F_1 and F_2, called the foci, is **constant**.

TI-Nspire Navigator Opportunity: *Quick Poll*
See Note 4 at the end of this lesson.

Teacher Tip: As students complete these questions, consider discussing the similarities and differences between ellipses and hyperbolas. One observation: a hyperbola is like "an ellipse turned inside out".

8. Click the sliders to set *c* = 5 and *d* = 8. Click and drag point *P* so that it lies on the x-axis at $(4,0)$ with $PF_1 = 9$ and $PF_2 = 1$.



a. The line segment of length 2*a* that has its endpoints at the vertices of the hyperbola is called the **transverse axis**. What is the length of the transverse axis?

<u>Sample Answers:</u> Since the coordinates of *O* are $(0,0)$ and the coordinates of *P* are $(4,0)$, $OP = 4$, and the length of the transverse axis is 8.

b. The line segment of length 2*b* that is perpendicular to the transverse axis at its center is called the **conjugate axis**. For a hyperbola, the lengths a, b, and c are related by the formula $c^2 = a^2 + b^2$. What is the length of the conjugate axis?

<u>Sample Answers:</u> We set the value *c* = 5 and, from part (a) above, we determined that *a* = 4. Thus, $5^2 = 4^2 + b^2$, and *b* = 3. Thus, the conjugate axis has a length 6.

9. The hyperbola has two branches that approach its linear asymptotes.

a. The two asymptotes will intersect at the center of the hyperbola. What point will lie on both asymptotes for the hyperbola on Page 2.2?

<u>Sample Answers:</u> The hyperbola on Page 2.2 has its center at the origin. Thus, the point $(0,0)$ will lie on both asymptotes.

b. Starting from the origin, as the asymptote increases by *b* units vertically, it will also increase *a* units horizontally. Represent the slope of one of the asymptotes in terms of *a* and *b*. What is the slope of the asymptote for the hyperbola on Page 2.2?

<u>Sample Answers:</u> The formula for the slope of a line is $m = \dfrac{\Delta y}{\Delta x}$. Since the slope increases *b* units vertically as it increases *a* units horizontally, the slope of one asymptote is $\dfrac{b}{a}$, in this case, $\dfrac{3}{4}$.

c. As the other asymptote decreases by *b* units vertically, it will also increase by *a* units horizontally. Represent the slope of the second asymptotes in terms of *a* and *b*. What is the slope of the second asymptote for the hyperbola on Page 2.2?

Sample Answers: Since the slope decreases *b* units vertically as it increases *a* units horizontally, the slope of that asymptote is $-\frac{b}{a}$, in this case, $-\frac{3}{4}$.

d. Write the equations for the two asymptotes of the hyperbola on Page 2.2.

Sample Answers: The equations for the two asymptotes are $y=\frac{3}{4}x$ and $y=-\frac{3}{4}x$.

e. Press ctrl G to open the function entry line, and enter your two equations for the asymptotes of the hyperbola. Check to see if your equations appear to be asymptotes for the hyperbola. If not, re-calculate and re-graph them.

10. The shape of a horizontal hyperbola is determined by its eccentricity, $e=\frac{c}{a}$, where *c* is the distance from the center of the hyperbola to a focus, and *a* is the horizontal distance from the center to a vertex. Use the slider to select an arbitrary value for d. Click on the *c*-slider, and notice what happens to the graph of the hyperbola as *c* gets larger and as *c* gets smaller. Use this information to give the range of values for *e*, the eccentricity of a hyperbola.

Sample Answers: As *c* gets larger, the foci move farther away from the vertices, and the hyperbola opens wider. As *c* gets smaller, the foci get closer to the vertices, and the hyperbola gets narrower. If *c* = *a*, the foci and vertices are the same points, so the hyperbola no longer exists. However, c can get infinitely large, making the hyperbola infinitely wide. Thus, $e>1$. Algebraically, since $0<a<c$, divide all terms by *a* to produce $0<1<\frac{c}{a}$. Since $e=\frac{c}{a}$, we see that $e>1$.

Wrap Up

Upon completion of the lesson, the teacher should ensure that students are able to understand:

- The foci definition of an ellipse.
- The foci definition of a hyperbola.
- The relationship between the location of the foci and the shapes of ellipses and hyperbolas.
- The effect of the eccentricity of ellipses and hyperbolas on the shape of their curves.

Assessment

Have students research real world applications of ellipses and hyperbolas to share with the class.

Enhancement

The standard form of the equation of an ellipse with a center at the origin can be derived from the definition of an ellipse and the distance formula, $\text{distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. Let $(\pm c, 0)$ represent the coordinates of the foci, $(\pm a, 0)$ represent the coordinates of the endpoints of the major axis (the vertices), $(0, \pm b)$ represent the coordinates of the endpoints of the minor axis (the co-vertices), and (x, y) represent a point on the ellipse. The sum of the distances from (x, y) is represented by $2a$.

TI-Nspire Navigator

Note 1
Name of Feature: Screen Capture and Live Presenter

You might want to take screen captures of students' screens as they drag point *P* and compare the locations of the various foci and lengths of PF_1 and PF_2. You might want to leave Screen Capture running in the background, with a 30 second automatic refresh, and without student names displayed, to enable you to monitor students' progress and make the necessary adjustments to your lesson.

You might want to ask one of the students to serve as the Live Presenter and demonstrate to the class how to navigate through the activity.

Note 2
Name of Feature: Quick Poll

You might want to take a Quick Poll to ensure that students understand the foci definition of an ellipse.

Note 3
Name of Feature: Screen Capture

You might want to take screen captures of students' screens as they drag point *P* and compare the locations of the various foci and lengths of PF_1 and PF_2.

Note 4
Name of Feature: Quick Poll

You might want to take a Quick Poll to ensure that students understand the foci definition of a hyperbola.